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FIBRATIONS WITH HOPFIAN PROPERTIES

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ABSTRACT

This study explores the homotopy-theoretic meeting-point of topics in differential topology, combinatorial group theory and algebraic K-theory. The first two are due to H. Hopf and date from around 1930. The third arose in the author's characterisation of plus-constructive fibrations. Let $F \xrightarrow{i} E \rightarrow B$ be a fibration such that i induces an isomorphism of homology with trivial integer coefficients; what is the effect of i on fundamental groups? In particular, when one passes to hypoabelianisations by factoring out perfect radicals, does i induce an epimorphism? Numerous conditions are determined which force an affirmative answer. On the other hand, negative examples of a non-finitary nature are also provided. This leaves the question open in the finitely generated case, where it forms a homological version of the dual to Hopf's original, famous question in group theory.

1. Introduction and history

The origins of this work lie in two research topics of Heinz Hopf, related in time as well as by author, but apparently mathematically unconnected until now. Each is well known, the former by differential topologists and the latter by combinatorial group-theorists.

1.1. [14] *Let $f: M_1 \rightarrow M_2$ be a smooth map between closed, connected orientable n-manifolds. If f induces an isomorphism of integral homology groups, then it induces an epimorphism of fundamental groups.*

Certain extensions beyond the smooth category are readily available. The

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question naturally prompted by these results concerns the most general category for which a map $f: X \to Y$ with $H_*(f)$ an isomorphism must have $\pi_1(f)$ surjective. (Homology is to be taken as having trivial integer coefficients.) The example of an embedding of any space X in its direct product with a non-contractible acyclic space shows that some modification is required (cf. [6]). Here we consider the surjectivity of the homomorphism $\pi_1(f)_{\mathcal{P}}$ defined in the following way.

Since the fundamental group of an acyclic space Z has trivial abelianisation $H_1(Z)$ it is a *perfect* group. Now any group G contains a unique maximal perfect subgroup (its *perfect radical*) $\mathcal{P}G$, the intersection of its transfinite derived series and so characteristic in G. The quotient $G/\mathcal{P}G$ is hypoabelian (that is, $\mathcal{P}(G/\mathcal{P}G) = 1$) and indeed the hypoabelian residual of G. By analogy with the term *abelianisation* for the abelian residual of a group, we may refer to $G/\mathcal{P}G$ as the *hypoabelianisation* $G_{\mathcal{P}}$ of G. As are derived groups and abelianisation, the perfect radical and hypoabelianisation are functorial — thus a group homomorphism $\phi: G \to H$ gives rise to $\mathcal{P}\phi: \mathcal{P}G \to \mathcal{P}H$ and $\phi_{\mathcal{P}}: G_{\mathcal{P}} \to H_{\mathcal{P}}$. Observe that when ϕ is the inclusion of G in its direct product with a perfect group, then the homomorphism $\phi_{\mathcal{P}}$ is just the identity.

1.2. [15] *Let N be a normal subgroup of a finitely generated group G. If G and G/N are isomorphic, must N be trivial?*

As a result of (1.2), groups isomorphic to no proper quotient are known as *Hopfian*. Two dual notions are feasible. The one, taking the dual of an epimorphism to be a monomorphism, is called the *co-Hopfian* property (see [17]). The other asks when a proper normal subgroup of a group G can be isomorphic to G; since our interest lies not in isomorphisms but in homology equivalences, this is the more appropriate dual, as the following example suggests.

EXAMPLE 1.3. (a) *If a finitely generated free group G admits a normal subgroup N such that the inclusion $\iota: N \hookrightarrow G$ induces an isomorphism $H_*(\iota)$, then $N = G$.*

(b) *However, the inclusion of the proper, non-normal subgroup* $\mathrm{Fr}(s, t^2s^{-1}t^{-1})$ *in the free group* $\mathrm{Fr}(s, t)$ *of rank 2 does induce a homology isomorphism.*

PROOF. (a) By [17 I.3.8, I.3.9, I.3.12], we have that the index of N in G is

$$|G:N| = \frac{\mathrm{rk}(N) - 1}{\mathrm{rk}(G) - 1},$$

where $\mathrm{rk}(N)$ denotes the rank of the free abelian group $H_1(N)$.

(b) On the other hand, since free groups admit no higher homology, any isomorphism of abelianisations is a homology isomorphism. The given example yields the automorphism of $\mathbf{Z} \oplus \mathbf{Z}$ corresponding to $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbf{Z})$. □

We again exclude the unwelcome influence of acyclic groups by passing to hypoabelianisations, and ask under what circumstances the following holds.

CONDITION 1.4. *If $N \overset{\iota}{\hookrightarrow} G \twoheadrightarrow G/N$ is a group extension with $H_*(\iota)$ an isomorphism, then $\iota_{\mathscr{P}}$ is an epimorphism.*

Since the classifying space functor $G \mapsto BG$ sends group extensions to fibrations, there is the following generalisation relating to (1.1) above.

CONDITION 1.5. *If $F \xrightarrow{i} E \to B$ is a fibration with $H_*(i)$ an isomorphism, then $\pi_1(i)_{\mathscr{P}}$ is an epimorphism.*

Thus (1.1) provides an example, (A) say, of a situation in which (1.5) holds. Matters related to (1.5) (when $F = E$) are discussed in [13]. All our other examples pertain more directly to (1.4). There is no loss of generality in restricting attention to (1.4), in the following sense.

PROPOSITION 1.6. *Condition 1.4 holds for all groups if and only if Condition 1.5 holds for all spaces.*

(Unless otherwise stated, all spaces, including fibres of fibrations, are assumed to be of the homotopy type of connected CW-complexes.) This result is proved at the end of Section 2 of this article. In its stated form, its force is diminished by the existence of examples (see Section 3 below) negating both conditions. However, it suggests that to each class of groups for which (1.4) holds there corresponds a class of spaces for which (1.5) is also true. The correspondence is given by theorems of Kan–Thurston type — see [6], [10].

Example (B) of the validity of (1.4), (1.5) is provided by (1.3)(a) above. Further evidence is as follows.

EXAMPLE (C). (1.4) holds if G is a finite group [9].

EXAMPLE (D). (1.4) holds if G is a nilpotent group [18], [19]; more generally, (1.5) holds if E (hence F [7]) is a nilpotent space [7].

EXAMPLES (E), (F). Further support for (1.4) is provided by S. Jackowski and Z. Marciniak of Warsaw (private communications) when G is respectively a cocompact or 1-relator group.

Further progress depends on a topological analysis of the situation, in the next section. This leads to new examples, generalising (B), (C) and (D) above, in §3. It is interesting (mathematically, as well as historically) to record how my interest in this problem came about. In characterising plus-constructive fibrations $F \to E \xrightarrow{p} B$ (those for which application of Quillen's plus-construction results in another fibration $F^+ \to E^+ \to B^+$) [3], I found the following two conditions to be necessary (and each, in favourable circumstances, sufficient).

CONDITIONS 1.7. (i) *$\mathscr{P}\pi_1(B)$ acts trivially on $H_*(F)$.*
(ii) *$\mathscr{P}\pi_1(p)$ is an epimorphism.*

(Condition (ii) is elsewhere described by the phrase "$\mathscr{P}\pi_1(p)$ is an epimorphism preserving perfect radicals".) Known examples of fibrations for which (ii) holds but not (i) include some of considerable significance for algebraic K-theory. However, none satisfying (i) without (ii) is readily available. This prompted the following.

PROBLEM 1.8. *In* (1.7), *does* (i) *imply* (ii)?

It turns out (Proposition 2.9 below) that this problem is intimately connected to our Condition 1.5.

2. Simplification

The following result, basic to our development, is doubtless quite well known. (Indeed, in our last conversation together, at the Berkeley ICM, Alex Zabrodsky outlined a spectral sequence proof.) However the proof below may have some novelty value. It is a pleasure to acknowledge the stimulation of an Oberwolfach conversation with M. Dyer in this connection.

LEMMA 2.1. *Suppose $k \geqq 1$. If the fibration $F \to E \to B$ has $H_i(F) \to H_i(E)$ an isomorphism for all $i \leqq k$ then both*
(i) *$\pi_1(B)$ acts trivially on $H_i(F)$ for all $i \leqq k$, and*
(ii) *$\tilde{H}_i(B) = 0$ for all $i \leqq k$.*
Conversely, (i) *and* (ii) *together imply that $H_i(F) \to H_i(E)$ is an isomorphism for all $i < k$ and an epimorphism for $i = k$.*

REMARKS 2.2. (a) In the presence of (ii), making $\pi_1(B)$ perfect, (i) is by [2] equivalent to the nilpotency of the action of $\pi_1(B)$.

(b) The bound stipulated in the converse is sharp; for example the classifying space fibration of the central extension $Z/2 \rightarrowtail SL(2, 5) \twoheadrightarrow PSL(2, 5)$ satisfies (i) and (ii) with $k = 1$, but the epimorphism $H_1(Z/2) \to H_1(SL(2, 5))$ fails to be an isomorphism.

PROOF. The sufficiency of (i) and (ii) is an immediate application of the Serre homology sequence. To prove their necessity, we recall that $\pi_1(E)$ acts trivially on $H_*(E)$ and thence, by injectivity, on $H_*(F)$. Then surjectivity of $\pi_1(E) \to \pi_1(B)$ implies that the action of $\pi_1(B)$ on $H_i(F)$ is also trivial. To obtain (ii), we first note that the case $i = 1$ follows from the usual exact sequence

$$H_0(B; H_1(F)) \to H_1(E) \to H_1(B)$$

whose former homomorphism here reduces to the epimorphism $H_i(F) \to H_i(E)$. Because $\pi_1(B)$ is thereby perfect, so that $\pi_1(B)_{\mathscr{P}} \cong \pi_1(pt)_{\mathscr{P}}$, we may apply to the map of fibrations

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \downarrow & & \downarrow \text{id} & & \downarrow \\ E & \xrightarrow{\text{id}} & E & \longrightarrow & pt \end{array}$$

the Serre spectral sequence comparison theorem [12 Theorem 3.5] as extended in [2]. This yields (ii). □

An attempt to generalise Lemma 2.1 to a a mod $\mathscr{C}$ version meets with unexpected subtleties.

PROPOSITION 2.3. *Let $\mathscr{C}$ denote a Serre class of abelian groups. Suppose that the fibration $F \xrightarrow{i} E \to B$ has $H_j(i)$ an isomorphism* mod *$\mathscr{C}$ for $j \leqq k$, where $k \leqq 3$. Then $H_j(B) \in \mathscr{C}$ for $j \leqq \min\{k, 2\}$ and, when $k = 3$, $H_3(B) \cong H_1(B; H_1(F))$* mod *$\mathscr{C}$. Moreover, when the fibration is the classifying space fibration for a group extension $N \hookrightarrow G \twoheadrightarrow G/N$, $H_1(Q; H_1(N))$ is isomorphic* mod *$\mathscr{C}$ to $H_1(N) \otimes_Q IQ$.*

PROOF. The proof involves chasing diagrams in the Serre spectral sequence, for which we use standard notation. We work mod $\mathscr{C}$, and in the diagrams linear sequences of maps are exact. First, from

$$
\begin{array}{ccccccc}
 & & H_1F & & & & \\
 & & \downarrow & \searrow^{H_1 i} & & & \\
H_2B & \xrightarrow{d^2} & H_0(B;H_1F) & \rightarrow & H_1E & \twoheadrightarrow & H_1B
\end{array}
$$

we have, since $H_1 i$ is mono, that

(a) $H_1F \rightarrow H_0(B; H_1F)$ *is iso*,

and

(b) $d^2 : H_2B \rightarrow H_0(B; H_1F)$ *is zero*,

and, since $H_1 i$ is epi, that

(c) $H_1B = 0$.

Second,

$$
\begin{array}{ccccc}
H_2F & & & & \\
\downarrow & \searrow^{0} & & & \\
H_2E & & & & \\
 & \searrow\!\!\!\searrow & & & \\
 & E^{\infty}_{2,0} & \rightarrowtail H_2B & \xrightarrow{d^2} & H_0(B; H_1F)
\end{array}
$$

implies

(d) $d^2 : H_2B \rightarrow H_0(B; H_1F)$ *is mono*.

From

$$
\begin{array}{ccccccc}
 & H_2E & & E^{\infty}_{1,1} & \twoheadleftarrow & H_1(B; H_1F) & \xleftarrow{d^2} H_3B \\
 & \nearrow & \nwarrow & \uparrow & & & \\
 & H_2F & & F_1 & & & \\
 & \downarrow\!\!\!\downarrow & & \rightarrowtail\!\!\!\!\uparrow & & & \\
E^3_{3,0} & \xrightarrow{d^3} E^3_{0,2} & \longrightarrow\!\!\!\!\rightarrow & E^{\infty}_{0,2}, & & &
\end{array}
$$

(e) $d^3 : E^3_{3,0} \rightarrow E^3_{0,2}$ *is zero*,

and

(f) $d^2 : H_3B \rightarrow H_1(B; H_1F)$ *is epi.*

Next, combining (e) above with

$$
\begin{array}{ccccc}
H_3F & & H_1(B; H_1F) & & \\
 & \searrow^{0} & \uparrow d^2 & & \\
\downarrow & & & & \\
H_3E & & H_3B & & \\
 & \searrow\!\!\!\searrow & \uparrow & & \\
 & E^{\infty}_{3,0} \rightarrowtail & E^3_{3,0} & \xrightarrow{d^3} & E^3_{0,2}
\end{array}
$$

gives

(g) $d^2 : H_3B \to H_1(B; H_1F)$ *is mono.*

Finally,

(h) $H_1(Q; H_1N) \to H_1(N) \otimes_Q IQ$ *is iso*

because of (a) and the classical exact sequence

$$H_1(Q; H_1N) \rightarrowtail H_1(N) \otimes_Q IQ \to H_1N \twoheadrightarrow H_0(Q; H_1N). \qquad \square$$

We now come to the result [4 (3.9), (3.8)] which translates Problem 1.8 into one reminiscent of Hopf's earlier work. Notation is given by the following diagram showing the map of fibrations generally obtained by applying the plus-construction to $F \xrightarrow{i} E \xrightarrow{p} B$.

$$\begin{array}{ccc} F & \xrightarrow{t} & F_{p^+} \\ {\scriptstyle i}\downarrow & & \downarrow \\ E & \longrightarrow & E^+ \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p^+} \\ B & \longrightarrow & B^+ \end{array}$$

LEMMA 2.4. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration with $H_1(B) = 0$.*

(i) *$\pi_1(B)$ acts trivially on $H_*(F)$ if and only if $H_*(t) : H_*(F) \to H_*(F_{p^+})$ is an isomorphism.*

(ii) *$\mathscr{P}\pi_1(p)$ is an epimorphism if and only if $\pi_1(t) : \pi_1(F) \to \pi_1(F_{p^+})$ is.* $\square$

REMARK 2.5. Again, there is an "up to dimension k" version of (2.4)(i); less predictably, a study of the proof of [4 (3.9)] reveals that each condition in (i) is equivalent to the injectivity of $H_i(t)$ ($i \leq k$). This fact leads immediately to the necessity of (i) in (2.1) because the isomorphism $H_i(F) \to H_i(E) \to H_i(E^+)$ factors through $H_i(t)$.

Since (2.1) dictates that the inclusion of the fibre induces a homology equivalence only in the presence of an acyclic base, we now consider a canonical method of passing to such a fibration.

$$
\begin{array}{ccc}
F & & F \\
{\scriptstyle \hat{\imath}}\downarrow & & \downarrow{\scriptstyle i} \\
\hat{E} & \longrightarrow & E \\
{\scriptstyle \hat{p}}\downarrow & & \downarrow{\scriptstyle p} \\
\mathscr{A}B & \longrightarrow & B
\end{array}
$$

In this pull-back diagram, $\mathscr{A}B \to B \to B^+$ is the acyclic fibration of Quillen and $\mathscr{A}$ Dror's acyclic functor as in, for example, [1 ch. 7]. We recall from [4 (3.7)] that, because $\mathscr{P}\pi_1(B^+) = 1$, $\mathscr{P}\pi_1(p)$ maps onto precisely when $\mathscr{P}\pi_1(\hat{p})$ does. Moreover, the assumption that $\pi_1(B)$ is perfect makes $\pi_1(\mathscr{A}B) \to \pi_1(B)$ epi; therefore $\pi_1(\mathscr{A}B)$ acts trivially on $H_*(F)$ if and only if $\pi_1(B)$ does. This fact combines with (2.1) (k infinite) to produce (i) of the following (in the above notation).

THEOREM 2.6. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration with $H_1(B) = 0$.*

(i) *$\pi_1(B)$ acts trivially on $H_*(F)$ if and only if $H_*(\hat{\imath})$ is an isomorphism.*

(ii) *If any of the three homomorphisms $\mathscr{P}\pi_1(p)$, $\pi_1(i)_{\mathscr{P}}$ and $\pi_1(\hat{\imath})_{\mathscr{P}}$ is an epimorphism, then so are the other two.*

PROOF. Only (ii) still requires proof. Since we have already deduced the equivalence of the surjectivity of $\mathscr{P}\pi_1(p)$ and $\mathscr{P}\pi_1(\hat{p})$, it remains to compare the former with $\pi_1(i)_{\mathscr{P}}$. We chase the following commutative diagram, noting that, since $\pi_1(B)$ is perfect, $\pi_1(B^+)$ is trivial and $\pi_1(F_{p^+}) \to \pi_1(E^+) = \pi_1(E)_{\mathscr{P}}$ is an epimorphism.

$$
\begin{array}{ccccc}
\pi_1(F) & & \xrightarrow{\pi_1(t)} & & \pi_1(F_{p^+}) \\
\downarrow & \searrow & & & \downarrow \\
{\scriptstyle \pi_1(i)} & & \pi_1(F)_{\mathscr{P}} & & \\
\downarrow & & {\scriptstyle \pi_1(i)_{\mathscr{P}}} & \searrow & \downarrow \\
\pi_1(E) & & \twoheadrightarrow & & \pi_1(E)_{\mathscr{P}}
\end{array}
$$

Clearly $\pi_1(i)_{\mathscr{P}}$ is epi provided $\pi_1(t)$ is. However, by (2.4)(ii) this is implied by (in fact equivalent to) $\mathscr{P}\pi_1(p)$ being epi.

A second chase of the above diagram establishes the converse whenever it is known that $\pi_1(F_{p^+}) \to \pi_1(E)_{\mathscr{P}}$ is an isomorphism. This happens when B^+ is 2-connected, in other words, when $H_1(B) = H_2(B) = 0$. We proceed to deduce the general case from this result. To do this, let S be the space labelled B_3 in

[1 (7.1)], namely the fibre of the map $B \to K(H_2(B), 2)$ that corresponds to the identity homomorphism on $H_2(B)$ under the isomorphism

$$\mathrm{Hom}(H_2(B), H_2(B)) \cong [B, K(H_2(B), 2)]$$

given by the universal coefficient formula (since $H_1(B) = 0$). We form the pull-back diagram of fibrations

$$\begin{array}{ccccc} & & F & & F \\ & & \downarrow & & \downarrow \\ K(H_2(B), 1) & \longrightarrow & R & \xrightarrow{r} & E \\ & & {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ K(H_2(B), 1) & \longrightarrow & S & \longrightarrow & B \end{array}$$

and observe that the general assertion follows immediately from the following three facts.

(a) $H_1(S) = H_2(S) = 0$;

(b) $\mathscr{P}\pi_1(p)$ *is epi if and only if* $\mathscr{P}\pi_1(q)$ *is epi*;

(c) $\pi_1(R)_{\mathscr{P}} \cong \pi_1(E)_{\mathscr{P}}$.

Now the first of these claims is proven in [1 (7.1)]. The second is an application of [4 (3.7)] to the above diagram. For (c), observe that because the kernel K of $\pi_1(r) : \pi_1(R) \twoheadrightarrow \pi_1(E)$ is central in $\pi_1(R)$, it lies in $\mathscr{P}\pi_1(R)$ and $\mathscr{P}\pi_1(r)$ is epi. So the extension $K \rightarrowtail \pi_1(R) \twoheadrightarrow \pi_1(E)$ restricts over $\mathscr{P}\pi_1(E)$ to the extension $K \rightarrowtail \mathscr{P}\pi_1(R) \twoheadrightarrow \mathscr{P}\pi_1(E)$. The stated isomorphism of quotients follows. □

REMARK 2.7. In (2.6), (i) remains valid with $\mathscr{A}B \to B$ replaced by any $\hat{B} \to B$ with $\hat{B}$ acyclic, so long as the image of $\pi_1(\hat{B})$ in $\pi_1(B)$ normally generates $\pi_1(B)$. On the other hand, in respect of (ii), for any map $f: \hat{B} \to B$, $\mathscr{P}\pi_1(p)$ is epi whenever $\mathscr{P}\pi_1(\hat{p})$ and $\mathscr{P}\pi_1(f)$ are; however the converse is more problematic (see the proof of [1 (3.7)]).

There is a group-theoretic counterpart to Theorem 2.6. It arises from the following analogue of Dror's acyclic space construction, introduced in [11] (for further refinements see [6]). Let Q be a perfect group. Then, as in [11 (5.7)], any epimorphism from a free group F_i to Q factors through the inclusion of F_i in the derived group F'_{i+1} of a further free group F_{i+1}. Then $\mathscr{F}Q = \bigcup F_i$ maps onto Q; since locally free, it shares with free groups the property that its homology in dimensions higher than 1 is trivial (because homology commutes with direct limits), and since by construction it is perfect it is therefore an acyclic group. We now consider the induced group extension

$$\begin{array}{ccccc} N & \xhookrightarrow{\bar{\iota}} & \bar{G} & \xrightarrow{\bar{\pi}} & \mathscr{F}Q \\ \| & & \downarrow & & \downarrow \\ N & \xhookrightarrow{\iota} & G & \xrightarrow{\pi} & Q \end{array}$$

The following result is an immediate application of Remark 2.7 to the above situation.

COROLLARY 2.8. *Let $N \overset{\iota}{\hookrightarrow} G \twoheadrightarrow Q$ be a group extension with Q perfect.*

(i) *Q acts trivially on $H_*(N)$ if and only if $H_*(\bar{\iota})$ is an isomorphism.*

(ii) *$\mathscr{P}\pi$ is an epimorphism if and only if $\iota_{\mathscr{P}}$ is, and each is an epimorphism if $\bar{\iota}_{\mathscr{P}}$ is.* □

We shall now combine (2.6) and (2.8) in order to establish the metatheorem linking Conditions 1.4 and 1.5 as well as Problem 1.8.

PROPOSITION 2.9. *The following are equivalent.*

(i) (1.4) *holds for all extensions $N \overset{\iota}{\hookrightarrow} G \twoheadrightarrow Q$.*

(ii) *If an extension $N \overset{\iota}{\hookrightarrow} G \overset{\pi}{\twoheadrightarrow} Q$ has $\mathscr{P}Q$ acting trivially on $H_*(N)$, then $\mathscr{P}\pi$ is an epimorphism.*

(iii) *If a fibration $F \overset{i}{\to} E \overset{p}{\to} B$ has $\mathscr{P}\pi_1(B)$ acting trivially on $H_*(F)$, then $\mathscr{P}\pi_1(p)$ is an epimorphism.*

(iv) (1.5) *holds for all fibrations $F \overset{i}{\to} E \to B$.*

PROOF. We argue in the direction (i)⇒(ii)⇒(iii)⇒(iv)⇒(i). As previously noted, (i) is just a special case of (iv), while the second implication is part of the statement of [6 (1.5)]. Next observe that we may assume that in (ii) (respectively (iii)) Q (resp. $\pi_1(B)$) is a perfect group. This is seen by taking the induced extension (resp. fibration) over $\mathscr{P}Q$ (resp. the covering of B associated to $\mathscr{P}\pi_1(B)$). To prove the first implication, let $N \overset{\iota}{\hookrightarrow} G \overset{\pi}{\twoheadrightarrow} Q$ be an extension with Q perfect and acting trivially on $H_*(N)$. So by Corollary 2.8(i) $H_*(\bar{\iota})$ is an isomorphism. From (1.4) $\bar{\iota}_{\mathscr{P}}$ is therefore surjective, whence (2.8)(ii) completes the implication. The final argument, showing (iii) implies (iv), proceeds similarly, using in turn (2.6)(i), (1.5) and (2.6)(ii). □

3. Further results

In this section we shall see how information obtained in §2 can be used to extend the list of examples of the validity of Conditions 1.4 and 1.5. First we may combine Theorem 2.6(ii) with [4 (2.3)], which lists various sufficient conditions for $\mathcal{P}\pi_1(p)$ to be surjective. (We denote the mth derived subgroup of a group G by $G^{(m)}$, so $G' = G^{(1)} = [G, G]$ and $G^m = G^{(m-1)\prime}$. Also, let $K = \operatorname{Im} \pi_1(i) = \operatorname{Ker} \pi_1(p) \leqq \pi_1(E)$.)

Example (G). (1.5) holds whenever $\pi_1(p)$ is split.

Example (H). (1.5) holds whenever $\pi_1(E)^{(m)} \leqq K \cdot \mathcal{P}\pi_1(E)$ for some finite m (and so whenever $\pi_1(E)$ is perfect-by-soluble). Note that this broadens Examples (G) and (D).

Example (I). (1.5) holds whenever $K^{(n)} \leqq \mathcal{P}\pi_1(E)$ for some finite n (and so whenever $\pi_1(F)$ is perfect-by-soluble).

Example (J). (1.5) holds whenever the homomorphism $\pi_1(E) \to \operatorname{Aut}(K/\mathcal{P}K)$, induced by conjugation, has hypoabelian image in $\operatorname{Out}(K/\mathcal{P}K) = \operatorname{Aut}(K/\mathcal{P}K)/\operatorname{Inn}(K/\mathcal{P}K)$.

Of course, in the situation of (1.4), K is just N, and Example (J) may be strengthened by the observation that it suffices to have hypoabelian image in $\operatorname{Out}_h(N) = \operatorname{Ker}[\operatorname{Out}(N) \to \operatorname{Aut}(H_*(N))]$. A consequence of this is the following, which relies on Theorems B and C of [8]. For the proof, see [6 (1.3)].

Example (K). (1.4) holds whenever N is itself an extension of a characteristic subgroup by a group of form F/R' where F is a non-cyclic finitely generated free group, $R \leqq F'$ is normal in F and F/R is residually torsion-free nilpotent. The special case where R is trivial and N is a finitely generated free group is Example (B).

Of the above examples, only (K) exploits the action on homology groups, and then only the first. Fuller use of this action requires recourse to (2.1) above and knowledge of the group-theoretic structure of acyclic groups imparted by [6]. The former of these examples shows that for $H_*(\imath)$ to be an isomorphism $G - N$ must contain relatively few elements of finite order.

Example (L). Suppose G admits a subgroup M of finite index which contains N but does not contain all elements of finite order in G. Then Q admits a subgroup of finite index which fails to contain all elements of finite order. By [6] such a Q cannot be acyclic, so that by (2.1) Condition 1.4 holds vacuously.

EXAMPLE (M). Suppose that G is generated by elements of finite order, and that Out(N) (or Out($N/\mathscr{P}N$)) admits a (subnormal) series whose factors are either torsion-free, hypoabelian or residually linear. (The last term means that any element of a factor group is mapped non-trivially by some linear complex representation of the factor.) By [6] and (2.1) again, the acyclic group Q must have trivial image in Out($N/\mathscr{P}N$), making $\iota_{\mathscr{P}}$ surjective as in Example (J).

Finally, we consider group extensions which violate Condition 1.4. One class of examples is presented in [6]. It relies on special properties of the derived length and generating sets for subgroups in finite nilpotent groups. Although groups obtained in this way can be countable, they are clearly infinitely generated. A more natural kind of example is the following, which is based on joint work of B. Hartley, M. Kuzucuożlu and V. Turau at Manchester. (I am grateful to them for permission to use this unpublished work.)

EXAMPLE 3.1. Let P be an arbitrary perfect group. As in §2 above, construct the acyclic group $Q = \mathscr{F}P = \bigcup F_i$ where each F_i ($i \in \mathbf{N}$) is free. Let N be the restricted direct product of the F_i, that is,

$$N = \{(x_1, x_2, \ldots) \in \Pi F_i \mid \exists h \text{ s.t. } \forall i \geqq h, x_i = 1\},$$

where ΠF_i stands for the unrestricted Cartesian product. Define also

$$G = \{(x_1, x_2, \ldots) \in \Pi F_i \mid \exists h \text{ s.t. } \forall i \geqq h, x_i = x_h\}.$$

Then the homomorphism $\pi : G \to Q$ that sends each $(x_1, x_2, \ldots)$ to its x_h is evidently an epimorphism with kernel N. Given $g \in G$ and a finite set $g'_1, \ldots, g'_k$ of elements of N, there is a number n such that no g'_j contains more than n non-trivial entries. Let g' be the element of N whose first n entries are those of g, with the remainder trivial. Then conjugation by g coincides with that by g' on the subgroup N_1 of N generated by $g'_1, \ldots, g'_k$; since it therefore corresponds to an inner automorphism of N, the image in $H_*(N)$ of its action on $H_*(N_1)$ is the identity. Now $H_*(N)$ is just the direct limit of such $H_*(N_1)$. So the action of Q on $H_*(N)$ is trivial. By (2.1) it follows that $\iota : N \hookrightarrow G$ induces an isomorphism in homology.

On the other hand, recall that free groups are residually nilpotent (e.g. [17 I.10.2]). That is, their normal subgroups affording nilpotent quotients have trivial intersection; equivalently, given non-trivial x_i in F_i, there is a nilpotent quotient $S(x_i)$ of F_i into which x_i maps non-trivially. Then g is also seen to be residually nilpotent because $(x_1, x_2, \ldots)$ maps non-trivially into the nilpotent quotient $\Pi\, S(x_i)$ of G. Thus G is hypoabelian and $\iota_{\mathscr{P}} = \iota$ is not an epimorphism.

An intriguing aspect of the above example is that it is no closer to being finitely generated than is that given in [6]. This prompts the following question.

QUESTION 3.2. *Is Condition* 1.4 *valid for all finitely generated groups* N, G *or* Q?

REFERENCES

1. A. J. Berrick, *An Approach to Algebraic K-Theory*, Research Notes in Math. 56, Pitman, London, 1982.
2. A. J. Berrick, *The plus-construction and fibrations*, Quart. J. Math. (2) **33** (1982), 149–157.
3. A. J. Berrick, *Characterisation of plus-constructive fibrations*, Adv. in Math. **48** (1983), 172–176.
4. A. J. Berrick, *Group epimorphisms preserving perfect radicals, and the plus-construction*, in *Algebraic K-Theory, Number Theory, Geometry and Analysis — Proc. Bielefeld 1982*, Lecture Notes in Math. **1046** Springer-Verlag, Berlin, 1984, pp. 1–12.
5. A. J. Berrick, *Remarks on the structure of acyclic groups*, in preparation.
6. A. J. Berrick and B. Hartley, *Perfect radicals and homology of group extensions*, Topology and its Applications **25** (1987), 65–73.
7. E. Dror, *A generalization of the Whitehead theorem*, Lecture Notes in Math. **249**, Springer-Verlag, Berlin, 1971, pp. 13–22.
8. J. L. Dyer and E. Formanek, *Characteristic subgroups and complete automorphism groups*, Am. J. Math. **99** (1977), 713–753.
9. L. Evens, *The cohomology ring of a finite group*, Trans. Am. Math. Soc. **101** (1961), 224–239.
10. J.-C. Hausmann, *Every finite complex has the homology of a duality group*, to appear.
11. A. Heller, *On the homotopy theory of topogenic groups and groupoids*, Illinois J. Math. **24** (1980), 576–605.
12. P. Hilton and J. Roitberg, *On the Zeeman comparison theorem for the homology of quasi-nilpotent fibrations*, Quart. J. Math. (2) **27** (1976), 433–444.
13. P. Hilton and J. Roitberg, *Note on epimorphisms and monomorphisms in homotopy theory*, Proc. Am. Math. Soc. **90** (1984), 316–320.
14. H. Hopf, *Zur Topologie der Abbildungen von Mannigfaltigkeiten, II*, Math. Ann. **102** (1930), 562–623.
15. H. Hopf, *Beiträge zur Klassifizierung der Flächenabbildungen*, J. Reine Angew. Math. **165** (1931), 225–236.
16. D. M. Kan and W. P. Thurston, *Every connected space has the homology of a* $K(\pi, 1)$, Topology **15** (1976), 253–258.
17. R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory*, Ergeb. der Math. 89, Springer-Verlag, Berlin, 1977.
18. J. R. Stallings, *Homology and central series of groups*, J. Algebra **2** (1965), 170–181.
19. U. Stammbach, *Anwendungen der Homologietheorie der Gruppen auf Zentralreihen und auf Invarianten von Präsentierungen*, Math. Z. **94** (1966), 157–177.